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The RPA equation embedded into infinite-dimensional Fock space F_∞*

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Abstract

To clear up both algebraic and geometric structures for integrable systems derived from self-consistent field theory, in particular, a geometrical aspect of the random phase approximation (RPA) equation is presented from the viewpoint of symmetry of the evolution equation. The RPA equation for an infinite-dimensional Grassmannian is constructed. It gives us a simple geometrical interpretation that the collective submanifold is a rotator on a curved surface.

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1. Introduction

The usual standard description of fermion many-body systems starts with the most basic approximation founded on the independent-particle picture, the self-consistent field (SCF) for motions of fermions. Hartree–Fock (HF) theory is one such approximation for ground states. Excited states are treated with the well-known random phase approximation (RPA) if only a small fluctuation in a time-dependent HF (TDHF) mean field is taken into account around a stationary HF ground-state solution [2]. The TDHF equation is a nonlinear equation owing to its SCF character and has no unique solution. Particle–hole pair operators of fermions with n single-particle states are closed under a Lie multiplication and form a basis of Lie algebra u_n [3]. The u_n Lie algebra generates a canonical transformation to a Slater determinant (S-det), the Thouless transformation [4], which induces a representation of the corresponding $U(n)$ group. It gives the $U(n)$ (HF) wavefunction of the independent-particle approximation. It also provides an exact generator coordinate representation of fermion state vectors in which

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the generating wavefunction is an independent-particle wavefunction. The RPA is a standard method to describe collective excitations and has been successful in explaining fermion systems with small quantum fluctuations.

In a recent series of papers [5–7], we have studied the relation between the TDHF theory [8] and the τ-functional method in soliton theory [9]. To go beyond a perturbative method with respect to periodic collective variables [10], we have aimed at constructing the SCF theory, i.e., the TDHF theory on the associative affine Kac–Moody algebra along the soliton theory on the infinite-dimensional fermions. They are introduced through the Laurent expansion of the finite-dimensional fermion operators with respect to degrees of freedom of the fermions related to the mean-field potential. We have attempted to embed the HF u_n Lie algebra into an infinite-dimensional Lie algebra gl_∞ with the aid of the Laurent expansion of the fermion operators with respect to the parameter z. Thus, the TDHF equation on the finite-dimensional Grassmannian Gr_m (m : number of hole states) is embedded into the infinite-dimensional Grassmannian. We have given an expression for the TDHF theory on the τ-functional space. We have also shown that the TDHF equation on an infinite-dimensional fermion Fock space F_∞ under level one is nothing other than the Laurent expansion of the TDHF equation on the Gr_m. The construction of the TDHF equation on F_∞ presents us explicit algebraic structures as a *gauge theory* inherent in SCF theory. From these facts, the SCF theory can be regarded as a method to determine self-consistently both quasi-particle energies and boson energies of collective motions which are unified into a *gauge phase*. Thus, we could obtain a common language, *the infinite-dimensional Grassmannian and the Lie algebra together with the associative affine Kac–Moody algebra*. They play important roles and become useful tools to discuss a relation between SCF theory and soliton theory on a group manifold.

The purpose of this paper is to give a geometrical aspect of the RPA equation [11, 12] and an explicit expression for the RPA equation with a normal mode on infinite-dimensional fermion Fock space F_∞, to clear up algebraic and geometric structures for integrable systems derived from SCF theory. We also discuss the relation between a *loop* collective path and a formal RPA equation. Consequently, it can be proved that the usual perturbative method with respect to periodic collective variables η and η^* in TDHF theory [10] is involved in the present method which aims to construct TDHF theory on the associative affine Kac–Moody algebra. It turns out that the collective submanifold is exactly a rotator on a curved surface in the infinite-dimensional Grassmannian. If we could arrive successfully at our final goal of clarifying the relation between SCF theory and soliton theory on a group, the present work may give us important clues for the description of large-amplitude collective motions in nuclei and molecules and for construction of multi-dimensional soliton equations [13, 14] since the collective motions usually occur in multi-dimensional *loop* space. In section 2, we show a simple geometrical aspect of the RPA equation. It is just a natural extension of the usual RPA equation of small amplitude around the ground state to any point on a collective submanifold which should be extracted. In section 3, a formal RPA equation is constructed on an infinite-dimensional Fock space F_∞. Finally, we will give a summary and some concluding remarks. In the appendices, we will reconstruct a particle–hole subgroup on F_∞ and embed an SCF Hamiltonian into F_∞.

2. Geometrical aspect of the RPA equation

Following [12], we first recapitulate the fundamental idea in the previous series of papers [11] for extraction of a collective submanifold out of a fully parametrized SCF group manifold. We study here only the case of TDHF theory. In viewing symmetries of time-evolution equations, let us consider an abstract evolution equation $\partial_t u(t) = K(u(t))$ for a function

u depending only on time t. Suppose that there exists a certain transformation which converts a solution for u to another solution. Introducing a parameter s different from t to specify such a solution, we assume another kind of evolution equation with respect to s, i.e., $\partial_s u(t,s) = \overline{K}\{u(t,s)\}$. Then, an integrability condition for existence of the transformation is given by $\partial_s K\{u(t,s)\} = \partial_t \overline{K}\{u(t,s)\}$.

The *method of maximal decoupling* [10], in which the *invariance principle of the Schrödinger equation* and *canonicity condition* play crucial roles, can be regarded as an extension of such a transformation to another transformation which is dependent on multiple group parameters of Lie groups of systems. In particular, the canonicity condition demands that collective variables η and η^* in TDHF must constitute an orthogonal coordinate system. To this transformation, we can give the following interpretation: based on the invariance principle and canonicity condition, transformation groups for truncating a collective submanifold of time and collective variables mean that the group parameters should constitute an orthogonal coordinate system. They, however, have not been used explicitly so as to manifest the importance of the integrability condition with respect to t and s.

In a differential geometrical approach to nonlinear problems, each of the integrability conditions is transcribed into zero curvature of the connection on the corresponding Lie groups of each system. Nonlinear evolution equations, e.g., soliton equations such as the KdV, KP, sine/sinh-Gordon equations, originate from the well-known Lax equation [15] which arises as zero curvature of the connection [16]. These soliton equations appear as Lie-algebra-valued evolution equations for *tangent vector fields* of local gauge fields depending on time t and space x coordinates. In contrast, in the ordinary TDHF SCF theory, the corresponding Lie groups are unitary groups which transform an orthonormal base of a system and are dependent on t but not on x.

Our basic idea lies in the introduction of a sort of Lagrange approach familiar to fluid dynamics to describe a collective coordinate system. This approach enables us to adopt a 1-form Ω which is linearly composed of a TDHF Hamiltonian and infinitesimal generators induced by collective-variable differentials of a canonical transformation $U(n)$. The curvature C of the system is defined as $C \stackrel{d}{=} \mathrm{d}\Omega - \Omega\wedge\Omega$ and the integrability condition reads $C = 0$. This condition expressed in the quasi-particle frame (QPF) is nothing but the formal RPA equation imposed by weak orthogonal conditions among the infinitesimal generators, i.e., an equation for *tangent vector fields* with respect to the collective variables on the group submanifold. Comparing with a theory for the local gauge fields, in our theory it must be noted that the degrees of freedom of the collective variables are involved in parameter space of the group manifold, which is quite different from an ordinary group manifold defined in a functional space of coordinate x and also time t. Overcoming this crucial difference, could we obtain a unified theoretical frame of an integrable system both in the SCF method and the soliton theory? If it is achieved, the present work becomes a powerful approach to a prescription for giving an answer to such an exciting problem.

Relative vector fields made of an SCF Hamiltonian around each point on any integral curves also constitute solutions for a formal RPA equation around the same point, which is in turn a fixed point in a QPF. This means that the formal RPA equation is a natural extension of the usual RPA equation for small-amplitude quantal fluctuations around a ground state to that at any point on a collective submanifold to be studied from now on. As an illustration of our theory, we show a simple geometrical aspect of the formal RPA equation (see figure 1).

In figure 1, the picture of solutions of the formal RPA equation is given with the help of a geometrical optics-like image on a curved surface on which the axis of time and that of collective-variable space are exchanged with each other. Suppose that there exist a coordinate

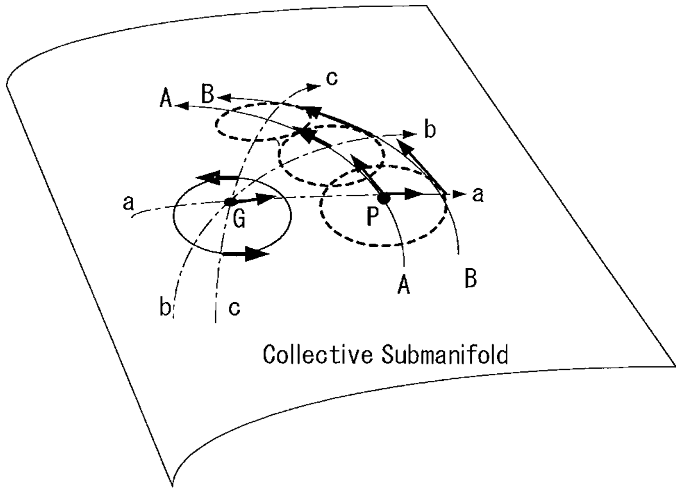

Figure 1. Formal RPA on the collective submanifold. G: a fixed point denoting a ground state and a usual RPA orbit; P: a fixed point and a formal RPA orbit on a moving frame; curves AA, BB: integral curves (big wave fronts); curves aa, bb, cc: collective coordinate (η).

system formed by a single pair of collective variables (η, $\eta^\star$) and a time t. The integral curves made of the SCF Hamiltonian draw big wave fronts. The trajectories by the formal RPA equation are drawn by small wave fronts occurring around a new fixed point which is on the big wave fronts. The enveloping curves made of the small wave fronts correspond to another big wave front. The time t carries one of group parameters for each point on each big wave front and the variables η and $\eta^\star$ cover other parameters of the group. If we put ourselves on a moving frame, the equation becomes a fundamental equation describing *tangent vector fields* fluctuating around us. It associates with the weak orthogonality conditions among the generating operators and causes an evolution in the space of the collective variables. Conversely speaking, the search for a solution of the equation leads us to a determination of the submanifold on which we really stand. As a consequence, the problem of extracting a certain collective submanifold out of the fully parametrized TDHF manifold may be reduced to the search for the corresponding sphere on which exists the top of an arrow attached to generators at a space around a fixed point. Then, it is interpreted that the formal RPA is just an extension of the usual RPA form on a flat surface (linear) to that on a curved surface (nonlinear). Furthermore, we note that the starting point selected by us on the moving frame becomes a standard point (new fixed point). This fact presents a geometrical interpretation for symmetry breaking and recovery. The former is brought as a choice of spontaneous symmetry breaking and the latter causes the motion, which has already been running, owing to a recovery of the symmetry. In other words, the equation of motion should be kept invariant if we select any coordinates. Through such an observation, it turns out that the formal RPA equation becomes an interesting illustration of a dynamical equation describing both the local and global SCF characters, i.e., the *tangent vector manifold* and the group manifold. With a wider viewpoint of the geometrical optics on curved manifolds, both local and global characters of the SCF theory can be argued parallelly to the case of the formal RPA equation in a similar way. At the same time the physical characteristics of the various Lie-algebra-valued operators are made easy to understand at any point on those manifolds.

The geometrical optic-like image leads to the following interpretation: the integrability condition (the formal RPA equation) is just the infinitesimal condition to transform a solution into another solution for the evolution equation under consideration. It is now easy to understand that the usual treatment of the RPA equation for small amplitude around a ground-state solution is nothing but a method of determining an infinitesimal transformation of symmetry under the assumption that fluctuating fields are composed only of normal modes. The case not satisfying such an assumption is exactly our main problem which is whether the fluctuating fields relate to the soliton theory or not. We intend to search for a framework suitable for approaching such a problem, which can simultaneously determine a certain fixed point and the collective submanifold connecting to it in the fully parametrized TDHF manifold. Then from this discussion, we may be convinced that it is natural for us to let the fixed point be dependent on the collective variables.

3. Construction of formal RPA equation on F_∞

Following [5, 6], we sketch briefly the TDHF method on F_∞. For fermion operators of n single-particle states in an almost time-periodic self-consistent mean-field potential with a normal mode ω_c, we introduce the infinite-dimensional fermion operators $\psi_{nr+\alpha}$ and $\psi^*_{nr+\alpha}$ ($\alpha = 1, \ldots, n, r \in \mathbb{Z}$), the normalized perfect vacuum $\{\psi_{nr+\alpha}|\mathrm{Vac}\rangle = 0, \langle\mathrm{Vac}|\psi^*_{nr+\alpha} = 0$ $(r \leqslant -1)$; $\psi^*_{nr+\alpha}|\mathrm{Vac}\rangle = 0, \langle\mathrm{Vac}|\psi_{nr+\alpha} = 0 (r \geqslant 0)\}$ with $\langle\mathrm{Vac}|\mathrm{Vac}\rangle = 1$ and the reference vacuum $\{|m\rangle = \psi_m \cdots \psi_1|\mathrm{Vac}\rangle, m = 1, \ldots, n\}$ having $\langle m|m\rangle = 1$. The normal-ordered pair operators $:\psi_{nr+\alpha}\psi^*_{ns+\beta}:$ $(\overset{d}{=} \psi_{nr+\alpha}\psi^*_{ns+\beta} - \delta_{\alpha\beta}\delta_{rs}(s < 0))$ generate an affine Kac–Moody algebra [17, 18]. We define the following $\widehat{su_n} (\subset \widehat{sl_n})$ Lie algebra:

$$
\left.\begin{array}{l}
X_\gamma = \widehat{X}_\gamma + \mathbb{C}\cdot c, \mathbb{C}^* = -\mathbb{C}, \text{(pure imaginary)} \\
\widehat{X}_\gamma = \sum_{r=-N}^{N} \sum_{s\in\mathbb{Z}} (\gamma_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \gamma_r^\dagger = -\gamma_{-r}, \mathrm{Tr}(\gamma_r) = 0, \\
[X_\gamma, c] = 0, [X_\gamma, X_{\gamma'}] = \widehat{X}_{[\gamma,\gamma']} + \alpha(\gamma,\gamma')\cdot c, c|m\rangle = 1\cdot|m\rangle,
\end{array}\right\} \tag{3.1}
$$

$$
\left.\begin{array}{l}
\gamma = \begin{array}{c} \uparrow \\ h \\ \downarrow \\ \uparrow \\ p \\ \downarrow \end{array}
\begin{array}{c}
\longleftarrow \quad h \quad \longrightarrow | \longleftarrow \quad p \quad \longrightarrow \\
\left[\begin{array}{ccccc|cccc}
 & & \ddots & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & & & \\
 & & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & & \ddots \\
\hline
\ddots & & & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & \\
 & \ddots & & & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 \\
 & & \ddots & & & & \ddots & &
\end{array}\right]
\end{array}, \\
\alpha(\gamma,\gamma') = \sum_{r=-N}^{N} r\mathrm{Tr}(\gamma_r\gamma'_{-r}) = -\frac{1}{2}\mathrm{Tr}\left[\begin{array}{ccc|ccc}
\ddots & & & & & \\
 & -I_n & & & & \\
 & & -I_n & & & \\
\hline
 & & & I_n & & \\
 & & & & I_n & \\
 & & & & & \ddots
\end{array}\right][\overline{\gamma}, \overline{\gamma}'],
\end{array}\right\} \tag{3.2}
$$

where c and I_n denote a centre and an n-dimensional unit matrix, respectively. The matrix γ is divided into four blocks by specifying apparently occupied states h and unoccupied states p for the perfect vacuum $|\mathrm{Vac}\rangle$. Corresponding to this division, a matrix in the 2-cocycle α is also divided into four blocks. The $\overline{\gamma}$ and $\overline{\gamma}'$ represent the off-diagonal parts of the matrices γ and γ' as

$$\overline{\gamma} \stackrel{d}{=} \left[\begin{array}{cc|cc} & & \gamma_2 & \ddots \\ & & \gamma_1 & \gamma_2 \\ \hline \gamma_{-2} & \gamma_{-1} & & \\ \ddots & \gamma_{-2} & & \end{array}\right], \qquad \overline{\gamma'} \stackrel{d}{=} \left[\begin{array}{cc|cc} & & \gamma'_2 & \ddots \\ & & \gamma'_1 & \gamma'_2 \\ \hline \gamma'_{-2} & \gamma'_{-1} & & \\ \ddots & \gamma'_{-2} & & \end{array}\right]. \tag{3.3}$$

Note the relation $\alpha^*(\gamma,\gamma') = -\alpha(\gamma,\gamma')$ and properties $\gamma^\dagger = -\gamma$ and $\gamma'^\dagger = -\gamma'$.

By a canonical transformation $U(\hat{g})(\hat{g} = \mathrm{e}^{\gamma})$, which satisfies the relations $U^{-1}(\hat{g}) = U(\hat{g}^{-1}) = U(\hat{g}^\dagger)$ and $U(\hat{g}\hat{g}') = U(\hat{g})U(\hat{g}')$ with $\hat{g}^\dagger\hat{g} = \hat{g}\hat{g}^\dagger = I_\infty$, the infinite-dimensional fermion operators are transformed into the forms

$$\psi_{nr+\alpha}(\hat{g}) \stackrel{d}{=} U(\hat{g})\psi_{nr+\alpha}U^{-1}(\hat{g}) = \sum_{s\in\mathbb{Z}} \psi_{n(r-s)+\beta}(g_s)_{\beta\alpha}, \tag{3.4}$$

together with their Hermitian conjugate. Here $I_\infty(=\hat{I})$ is an infinite-dimensional unit matrix and

$$\hat{g}_{nr+\alpha,ns+\beta} = (g_{s-r})_{\alpha\beta}, \qquad \hat{g}^\dagger_{nr+\alpha,ns+\beta} = \left(g^\dagger_{r-s}\right)_{\alpha\beta}, \tag{3.5}$$

$$\delta_{rs}\delta_{\alpha\beta} = (\hat{g}\hat{g}^\dagger)_{nr+\alpha,ns+\beta} = \sum_{t\in\mathbb{Z}} \left(g_t g^\dagger_{t+(r-s)}\right)_{\alpha\beta}, \tag{3.6}$$

together with the same relation for $(\hat{g}^\dagger\hat{g})_{nr+\alpha,ns+\beta}$. Note that $\hat{g}$ forms a periodic sequence with period n and formally s and t run over a infinite set of $\mathbb{Z}$.

The elements of the density matrix, corresponding to the formal Laurent expansion of the usual one on the finite-dimensional Grassmannian Gr_m, can be defined as

$$\begin{aligned}(W_r)_{\alpha\beta} &\stackrel{d}{=} \sum_{s\in\mathbb{Z}} \langle m|U(\hat{g}^\dagger) : \psi_{n(s+r)+\beta}\psi^*_{ns+\alpha} : U(\hat{g})|m\rangle \\ &= \sum_{s\in\mathbb{Z}}\sum_{\gamma=1}^{m} (g_s)_{\alpha\gamma}\left(g^\dagger_{s-r}\right)_{\gamma\beta}.\end{aligned} \tag{3.7}$$

Following [5, 6], we can obtain the SCF Hamiltonian on F_∞ as

$$\left.\begin{aligned} H_{F_\infty;\mathrm{HF}} &= \sum_{k\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} (\mathcal{F}_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \\ (\mathcal{F}_r)_{\alpha\beta} &\stackrel{d}{=} h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma}. \end{aligned}\right\} \tag{3.8}$$

Let us recapitulate briefly the previous main results [5, 6]. Assuming $z = \mathrm{e}^{-\mathrm{i}\omega_c t}(\hbar = 1)$ and then using a covariant differential operator $D_r \stackrel{d}{=} \mathrm{i}\partial_t + r\omega_c$, one can express the TDHF equation for $\hat{g}$ as

$$\left.\begin{aligned}
&D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g}, \qquad D_t\hat{g} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & D_{-1}g_{-1} & D_0g_0 & D_1g_1 & & \\ & & D_{-1}g_{-1} & D_0g_0 & D_1g_1 & \\ & & & D_{-1}g_{-1} & D_0g_0 & D_1g_1 \\ \ddots & & & & & \ddots \end{bmatrix}, \\
&\mathcal{F}(\hat{g}) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & & \\ & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & \\ & & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 \\ \ddots & & & & & \ddots \end{bmatrix}, \qquad \hat{g} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & g_{-1} & g_0 & g_1 & & \\ & & g_{-1} & g_0 & g_1 & \\ & & & g_{-1} & g_0 & g_1 \\ \ddots & & & & & \ddots \end{bmatrix}.
\end{aligned}\right\} \tag{3.9}$$

Upon the introduction of $\left(\mathcal{F}_r^c\right)_{\alpha\beta}(\hat{g}, \omega_c) \stackrel{d}{=} \omega_c \sum_{s\in\mathbb{Z}} s\left(g_s g_{s-r}^\dagger\right)_{\alpha\beta}$, the matrix $\mathcal{F}^c(\hat{g}, \omega_c)$ takes the form

$$\begin{aligned}
\mathcal{F}^c(\hat{g}, \omega_c) = \omega_c &\begin{bmatrix} \ddots & & & & & \ddots \\ & -g_{-1} & 0 & g_1 & & \\ & & -g_{-1} & 0 & g_1 & \\ & & & -g_{-1} & 0 & g_1 \\ \ddots & & & & & \ddots \end{bmatrix} \\
\times &\begin{bmatrix} \ddots & & & & & \ddots \\ & g_1^\dagger & g_0^\dagger & g_{-1}^\dagger & & \\ & & g_1^\dagger & g_0^\dagger & g_{-1}^\dagger & \\ & & & g_1^\dagger & g_0^\dagger & g_{-1}^\dagger \\ \ddots & & & & & \ddots \end{bmatrix}.
\end{aligned} \tag{3.10}$$

Then, equation (3.9) transforms into

$$\left.\begin{aligned}
&\mathrm{i}\partial_t\hat{g} = \mathcal{F}^p(\hat{g})\hat{g}, \qquad \mathcal{F}^p(\hat{g}) \stackrel{d}{=} \mathcal{F}(\hat{g}) - \mathcal{F}^c(\hat{g}), \\
&\left(\mathcal{F}_r^p\right)_{\alpha\beta} \stackrel{d}{=} \left(\mathcal{F}_r - \mathcal{F}_r^c\right)_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma} - \omega_c \sum_{s\in\mathbb{Z}} s\left(g_s g_{s-r}^\dagger\right)_{\alpha\beta},
\end{aligned}\right\} \tag{3.11}$$

introducing $\widehat{D}_t \stackrel{d}{=} \mathrm{i}\partial_t + H^c_{F_\infty;\mathrm{HF}}$, which is recast into that on the state vector $U(\hat{g})|m\rangle$ as

$$\left.\begin{aligned}
&\widehat{D}_t U(\hat{g})|m\rangle = H_{F_\infty;\mathrm{HF}} U(\hat{g})|m\rangle, \qquad H^c_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}} \left(\mathcal{F}_r^c\right)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \\
&\mathrm{i}\partial_t U(\hat{g})|m\rangle = H^p_{F_\infty;\mathrm{HF}} U(\hat{g})|m\rangle, \qquad H^p_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}} \left(\mathcal{F}_r^p\right)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : .
\end{aligned}\right\} \tag{3.12}$$

From now we will show how to embed the usual RPA equation on Gr_m into the F_∞. Suppose that $\hat{g}$ and $U(\hat{g})|m\rangle$ diagonalize $\mathcal{F}^p$ in $H^p_{F_\infty;\mathrm{HF}}$ and $\mathcal{F}^c$ in $H^c_{F_\infty;\mathrm{HF}}$, respectively. They are determined spontaneously if conditions $\hat{g} \simeq \hat{g}^0\,\mathrm{e}^{-\mathrm{i}\hat{\epsilon}t}$ and $\partial_t\hat{g}^0 = 0$ are satisfied. Using

(3.10), we have $\omega_c \Gamma(\hat{g}^0) = \mathcal{F}(\hat{g}^0)\hat{g}^0 - \hat{g}^0\hat{\epsilon}$, where $\Gamma(\hat{g}^0)$ is defined by

$$\Gamma(\hat{g}^0) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & & \ddots \\ & -g_{-1}^0 & 0 & g_1^0 & & & \\ & & -g_{-1}^0 & 0 & g_1^0 & & \\ & & & -g_{-1}^0 & 0 & g_1^0 & \\ \ddots & & & & & & \ddots \end{bmatrix}, \qquad \hat{\epsilon} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & \\ & \epsilon & & & \\ & & \epsilon & & \\ & & & \epsilon & \\ & & & & \ddots \end{bmatrix}. \tag{3.13}$$

The $\epsilon\,(\epsilon_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta})$ is a quasi-particle energy and g_r^0 is given by $g_r^0 \cdot z^r \propto \mathrm{e}^{-\mathrm{i}(\epsilon + \omega_c r I_n)t}$. Thus, the quasi-particle energy ϵ and boson energy ω_c are unified into a *gauge phase*. In the usual static HF theory on the Gr_m, the term related to a collective motion in (3.13) cannot evidently exist. Then, we must put $\omega_c \Gamma(\hat{g}^0) = 0$ and the $\hat{g}^0$ should be composed from only a block-diagonal g_0^0 given by $g_0^0 = \exp \gamma_0$ ($\gamma_0 \in su_n$; block-diagonal matrix of the γ (3.2)). The usual RPA equation around the static solution is realized exactly by the approximation made below

$$\hat{g} \simeq \hat{g}^0 \hat{\tilde{g}}\, \mathrm{e}^{-\mathrm{i}\hat{\epsilon}t}, \qquad \hat{\tilde{g}} = \exp \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ & \tilde{\gamma}_{-1} & 0 & \tilde{\gamma}_1 & & & \\ & & \tilde{\gamma}_{-1} & 0 & \tilde{\gamma}_1 & & \\ & & & \tilde{\gamma}_{-1} & 0 & \tilde{\gamma}_1 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.14}$$

where $\hat{\epsilon} = \hat{g}^{0\dagger}\mathcal{F}(\hat{g}^0)\hat{g}^0$ and $\partial_t \hat{g}^0 = 0$. The norm of $\tilde{\gamma}$ is very small, $\|\tilde{\gamma}\| \approx 0$. Here we have used the phase equivalence $U(\mathrm{e}^{-\mathrm{i}\hat{\epsilon}t})|m\rangle = \mathrm{e}^{-\mathrm{i}\sum_{\alpha=1}^m \varepsilon_\alpha t}|m\rangle \simeq |m\rangle$, i.e., $\hat{g}^0\,\mathrm{e}^{-\mathrm{i}\epsilon t} \simeq \hat{g}$.

We are now at a stage to construct the formal RPA equation on F_∞. Let ϵ and ϵ^* be parameters drawing a two-dimensional surface made of a continuous deformation of *loop* path on the Gr_m. Put $z = \mathrm{e}^{\mathrm{i}\varphi}$ and $\varphi = -\omega_c t$ and let the parameters to be independent on angle φ on each *loop*, namely, $\gamma = \sum_{r=-N}^N \gamma_r(\epsilon, \epsilon^*) z^r$ and $\frac{\partial \gamma_r}{\partial \varphi} = 0$ for all r, as is shown later. Then, as a whole we can identify ϵ and ϵ^* with collective variables η and η^* [5, 6]. A local representation on the surface is nothing but the integrability condition which is expressed with the use of the previous framework of formal RPA equation [11]. We put the following canonicity conditions which guarantee the variables (ϵ, ϵ^*) to be an orthogonally canonical coordinate system [8, 10, 6]:

$$\left.\begin{aligned} \langle \hat{g}|\partial_\epsilon|\hat{g}\rangle &\stackrel{d}{=} \langle m|U(\hat{g}^\dagger)\partial_\epsilon U(\hat{g})|m\rangle = \tfrac{1}{2}\epsilon^*, \\ \langle \hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle &\stackrel{d}{=} \langle m|U(\hat{g}^\dagger)\partial_{\epsilon^*} U(\hat{g})|m\rangle = -\tfrac{1}{2}\epsilon. \end{aligned}\right\} \tag{3.15}$$

We define infinitesimal generators on the collective submanifold as follows:

$$\left.\begin{aligned} &X_{\theta^\dagger} \stackrel{d}{=} \mathrm{i}\partial_\epsilon U(\hat{g}) \cdot U(\hat{g})^\dagger = \overline{X}_{\theta^\dagger} + \mathbb{C}(\mathrm{i}\partial_\epsilon \hat{g} \cdot \hat{g}^\dagger), \qquad \theta^\dagger \stackrel{d}{=} \mathrm{i}\partial_\epsilon \hat{g} \cdot \hat{g}^\dagger, \\ &X_{\theta} \stackrel{d}{=} \mathrm{i}\partial_{\epsilon^*} U(\hat{g}) \cdot U(\hat{g})^\dagger = \overline{X}_{\theta} + \mathbb{C}(\mathrm{i}\partial_{\epsilon^*} \hat{g} \cdot \hat{g}^\dagger), \qquad \theta \stackrel{d}{=} \mathrm{i}\partial_{\epsilon^*} \hat{g} \cdot \hat{g}^\dagger, \\ &\theta^\dagger = \begin{bmatrix} \ddots & & & & & \ddots \\ & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & & \\ & & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & \\ & & & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger \\ \ddots & & & & & \ddots \end{bmatrix}, \quad \theta = \begin{bmatrix} \ddots & & & & & \ddots \\ & \theta_{-1} & \theta_0 & \theta_1 & & \\ & & \theta_{-1} & \theta_0 & \theta_1 & \\ & & & \theta_{-1} & \theta_0 & \theta_1 \\ \ddots & & & & & \ddots \end{bmatrix}, \end{aligned}\right\} \tag{3.16}$$

where terms $\mathbb{C}(\cdots)$ vanish as was proved in [5, 6]. From $\partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle$, we obtain the *weak* orthogonality condition

$$
\begin{aligned}
1 &= \langle\hat{g}|[X_\theta, X_{\theta^\dagger}]|\hat{g}\rangle \\
&= \sum_{\alpha=1}^{m}\sum_{\gamma=1}^{n}\sum_{r\in\mathbb{Z}}([\theta,\theta^\dagger]_r)_{\alpha\gamma}(W_{-r})_{\gamma\alpha} - \frac{1}{2}Tr\begin{bmatrix}-\hat{I} & \\ & \hat{I}\end{bmatrix}[\bar{\theta},\bar{\theta}^\dagger],
\end{aligned}
\tag{3.17}
$$

where we have used equations (3.1) and (3.7).

As discussed in [6], using the idea of Lax pairs [15] we can recast equations (3.9) and (3.16), respectively, into

$$
\left.
\begin{aligned}
&D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g}, && \partial_t\hat{g}^0 = 0, && \mathcal{F}(\hat{g}) = \mathcal{F}(\hat{g}^0), \\
&\mathrm{i}\partial_\epsilon\hat{g} = \theta^\dagger(\hat{g})\hat{g}, && \theta^\dagger(\hat{g}) = \theta^\dagger(\hat{g}^0) + \hat{g}^0(\partial_\epsilon\hat{\epsilon})\hat{g}^{0\dagger}\cdot t, \\
&\mathrm{i}\partial_{\epsilon^*}\hat{g} = \theta(\hat{g})\hat{g}, && \theta(\hat{g}) = \theta(\hat{g}^0) + \hat{g}^0(\partial_{\epsilon^*}\hat{\epsilon})\hat{g}^{0\dagger}\cdot t.
\end{aligned}
\right\}
\tag{3.18}
$$

Upon introduction of $E = \sum_{\alpha=1}^{m}\epsilon_\alpha(\epsilon,\epsilon^*)$, the canonity condition (3.15) transforms into

$$
\left.
\begin{aligned}
\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle &= \langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle - \mathrm{i}\partial_\epsilon E\cdot t = \tfrac{1}{2}\epsilon^* - \mathrm{i}\partial_\epsilon E\cdot t, \\
\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle &= \langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle - \mathrm{i}\partial_{\epsilon^*}E\cdot t = -\tfrac{1}{2}\epsilon - \mathrm{i}\partial_{\epsilon^*}E\cdot t.
\end{aligned}
\right\}
\tag{3.19}
$$

From equation (3.19), the *weak* orthogonality condition (3.17) is expressed as

$$
\begin{aligned}
1 &= \partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle \\
&= \partial_{\epsilon^*}\langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle - \partial_\epsilon\langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle = \langle\hat{g}^0|[X_{\theta(\hat{g}^0)}, X_{\theta^\dagger(\hat{g}^0)}]|\hat{g}^0\rangle.
\end{aligned}
\tag{3.20}
$$

To satisfy integrability conditions for ϵ, ϵ^* and t, curvatures obtained from (3.18) should vanish; that is,

$$
\left.
\begin{aligned}
\mathcal{C}_{t,\epsilon} &\stackrel{d}{=} D_t\theta^\dagger(\hat{g}) - \mathrm{i}\partial_\epsilon\mathcal{F}(\hat{g}) + [\theta^\dagger(\hat{g}), \mathcal{F}(\hat{g})] = 0, \\
\mathcal{C}_{t,\epsilon^*} &\stackrel{d}{=} D_t\theta(\hat{g}) - \mathrm{i}\partial_{\epsilon^*}\mathcal{F}(\hat{g}) + [\theta(\hat{g}), \mathcal{F}(\hat{g})] = 0, \\
\mathcal{C}_{\epsilon,\epsilon^*} &\stackrel{d}{=} \mathrm{i}\partial_\epsilon\theta(\hat{g}) - \mathrm{i}\partial_{\epsilon^*}\theta^\dagger(\hat{g}) + [\theta(\hat{g}), \theta^\dagger(\hat{g})] = 0,
\end{aligned}
\right\}
\tag{3.21}
$$

and $\partial_t\hat{g}^0 = 0$. Here, $D_t\theta$ and $D_t\theta^\dagger$ are defined as

$$
(D_t\theta)_r = D_{r;t}\theta_r = (\mathrm{i}\partial_t + r\omega_c)\theta_r, \qquad (D_t\theta^\dagger)_r = D_{r;t}\theta^\dagger_{-r} = (\mathrm{i}\partial_t + r\omega_c)\theta^\dagger_{-r}.
\tag{3.22}
$$

The expressions for the curvatures on the quasi-particle frame (QPF) are the same forms as those of RPA equations in the finite Fock space [11]. As mentioned before, the TDHF equation on the F_∞ leads to the RPA equation if we take into account only a small fluctuation around a stationary ground-state solution. The form of RPA equation on the QPF has a following simple geometrical interpretation: relative vector fields made of the SCF Hamiltonian around each point on *loop* paths also take the form of RPA equation around the same point which is in turn a fixed point in the QPF. Thus, the curvature equation in the QPF is regarded as the formal RPA equation on the infinite-dimensional Grassmannian. Using (3.4), the canonical transformation for $\hat{g}$ is given by

$$
\psi_{nr+\alpha}(\hat{g}) = \sum_{s\in\mathbb{Z}}\sum_{\beta=1}^{n}\psi_{n(r-s)+\beta}\left(g_s^0\right)_{\beta\alpha}\mathrm{e}^{-\mathrm{i}\epsilon_\alpha t},
\tag{3.23}
$$

together with its Hermitian conjugate. According to [11], equation (3.18) is rewritten on the above QPF as

$$
\left.
\begin{aligned}
&-D_t\hat{g}^\dagger = \mathcal{F}(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, && \mathcal{F}(\hat{g}^\dagger)|_{\mathrm{qpf}} \stackrel{d}{=} \hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g}, \\
&-\mathrm{i}\partial_\epsilon\hat{g}^\dagger = \theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, && \theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}} \stackrel{d}{=} \hat{g}^\dagger\theta^\dagger(\hat{g})\hat{g}, \\
&-\mathrm{i}\partial_{\epsilon^*}\hat{g}^\dagger = \theta(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, && \theta(\hat{g}^\dagger)|_{\mathrm{qpf}} \stackrel{d}{=} \hat{g}^\dagger\theta(\hat{g})\hat{g},
\end{aligned}
\right\}
\tag{3.24}
$$

$$\theta^\dagger|_{\mathrm{qpf}} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & & & \\ & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & & \\ & & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & \\ \ddots & & & & & \ddots \end{bmatrix}_{\mathrm{qpf}},$$

$$\theta|_{\mathrm{qpf}} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ \theta_{-1} & \theta_0 & \theta_1 & & & \\ & \theta_{-1} & \theta_0 & \theta_1 & & \\ & & \theta_{-1} & \theta_0 & \theta_1 & \\ \ddots & & & & & \ddots \end{bmatrix}_{\mathrm{qpf}}. \tag{3.25}$$

The subscript 'qpf' means the quasi-particle frame (QPF). For equation (3.21), we obtain also another expression on this QPF as

$$\left.\begin{array}{ll} (D_t\theta^\dagger - \mathrm{i}\partial_\epsilon\mathcal{F} - [\theta^\dagger, \mathcal{F}])|_{\mathrm{qpf}} = 0, & (D_t\theta - \mathrm{i}\partial_{\epsilon^*}\mathcal{F} - [\theta, \mathcal{F}])|_{\mathrm{qpf}} = 0, \\ (\mathrm{i}\partial_\epsilon\theta - \mathrm{i}\partial_{\epsilon^*}\theta^\dagger - [\theta, \theta^\dagger])|_{\mathrm{qpf}} = 0. & \end{array}\right\} \tag{3.26}$$

Further, using (3.24) and the relation $\mathrm{i}\partial_\epsilon\mathcal{F}|_{\mathrm{qpf}} = \mathrm{i}\partial_\epsilon(\hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g}) = -[\theta^\dagger, \mathcal{F}]|_{\mathrm{qpf}} + \hat{g}^\dagger\mathrm{i}\partial_\epsilon\mathcal{F}\hat{g}$, one can rewrite equations in the first line of (3.26) as

$$D_t\theta^\dagger|_{\mathrm{qpf}} - \hat{g}^\dagger\mathrm{i}\partial_\epsilon\mathcal{F}(\hat{g})\hat{g} = 0, \qquad D_t\theta|_{\mathrm{qpf}} - \hat{g}^\dagger\mathrm{i}\partial_{\epsilon^*}\mathcal{F}(\hat{g})\hat{g} = 0. \tag{3.27}$$

From equations (3.25) and (3.18), the infinitesimal operators are expressed as

$$\theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}} = -\mathrm{i}\partial_\epsilon\hat{g}^\dagger\cdot\hat{g} = \mathrm{e}^{\mathrm{i}\hat{\epsilon}t}\{\partial_\epsilon\hat{\epsilon}\cdot t + \theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\}\,\mathrm{e}^{-\mathrm{i}\hat{\epsilon}t}, \tag{3.28}$$

together with the same relation for $\theta(\hat{g}^\dagger)|_{\mathrm{qpf}}$. We have also $\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}} = -\mathrm{i}\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0$ and $\theta(\hat{g}^{0\dagger})|_{\mathrm{qpf}} = -\mathrm{i}\partial_{\epsilon^*}\hat{g}^{0\dagger}\cdot\hat{g}^0$. Then, from (3.27) we can derive the formal RPA equation on the infinite-dimensional Grassmannian in the form

$$\left.\begin{array}{l} \omega_c\Gamma\{\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\} + \mathrm{i}\partial_\epsilon\hat{\epsilon} - [\hat{\epsilon}, \theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}] - i\hat{g}^{0\dagger}\partial_\epsilon\mathcal{F}(\hat{g}^0)\hat{g}^0 = 0, \\ \Gamma\{\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} & & & \\ & -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} & & \\ & & -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} & \\ \ddots & & & & & \ddots \end{bmatrix}_{\mathrm{qpf}}, \end{array}\right\} \tag{3.29}$$

and h.c. We had attempted to solve the formal RPA equation on an $SO(2n)$ group by means of the Taylor expansion with respect to the collective variables [11]. We, however, had not become aware of equation (3.29) on the infinite-dimensional Lie algebra. To obtain an explicit expression for the last term of the lhs of the first line in (3.29), we introduce an auxiliary density matrix $\widehat{R} = \hat{g}^0 \widehat{I}_{m\otimes(n-m)} \hat{g}^{0\dagger}$, where

$$\widehat{I}_{m\otimes(n-m)} \stackrel{d}{=} \begin{bmatrix} \ddots & & & \\ & I_{m\otimes(n-m)} & & \\ & & I_{m\otimes(n-m)} & \\ & & & I_{m\otimes(n-m)} \\ & & & & \ddots \end{bmatrix}, \qquad I_{m\otimes(n-m)} \stackrel{d}{=} \begin{bmatrix} -I_m & \\ & I_{n-m} \end{bmatrix}. \tag{3.30}$$

Then, using equation (3.5) the auxiliary density matrix $\widehat{R}$ is expressed as

$$\widehat{R} = \begin{bmatrix} & \ddots & & & & \ddots \\ & R_{-1} & R_0 & R_1 & & \\ & & R_{-1} & R_0 & R_1 & \\ & & & R_{-1} & R_0 & R_1 \\ \ddots & & & & \ddots & \end{bmatrix}, \qquad R_r \stackrel{d}{=} \sum_{s\in\mathbb{Z}} g_s^0 I_{m\otimes(n-m)} g_{s-r}^{0\dagger}. \tag{3.31}$$

Let us recall that $\widehat{I}$ is the infinite-dimensional unit matrix. Then, the $\widehat{R}$ is related to the density matrix $\widehat{W}$ as $\widehat{R} = \widehat{I} - 2\widehat{W}$ where

$$\widehat{W} = W(\hat{g}) \stackrel{d}{=} \begin{bmatrix} & \ddots & & & & \ddots \\ & W_{-1} & W_0 & W_1 & & \\ & & W_{-1} & W_0 & W_1 & \\ & & & W_{-1} & W_0 & W_1 \\ \ddots & & & & \ddots & \end{bmatrix}. \tag{3.32}$$

Then, we obtain

$$\begin{aligned} \mathrm{i}\partial_\epsilon \widehat{W} &= -\tfrac{1}{2}\hat{g}^0\{-\mathrm{i}\partial_\epsilon \hat{g}^{0\dagger}\cdot\hat{g}^0\widehat{I}_{m\otimes(n-m)} - \widehat{I}_{m\otimes(n-m)}(-\mathrm{i}\partial_\epsilon \hat{g}^{0\dagger}\cdot\hat{g}^0)\}\hat{g}^{0\dagger} \\ &= -\tfrac{1}{2}\hat{g}^0[\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}, \widehat{I}_{m\otimes(n-m)}]\hat{g}^{0\dagger}, \end{aligned} \tag{3.33}$$

and h.c. In the above, we have used equations (3.28) and (3.31). Further, we introduce the following quantities:

$$\left.\begin{aligned} &\theta_r^{0\dagger}\big|_{\mathrm{qpf}} \stackrel{d}{=} \begin{bmatrix} \xi_r^0 & \phi_r^0 \\ \psi_r^0 & \bar{\xi}_r^0 \end{bmatrix}, \qquad B_r^\dagger\big|_{\mathrm{qpf}} \stackrel{d}{=} -\frac{1}{2}\left[\theta_r^{0\dagger}\big|_{\mathrm{qpf}}, I_{m\otimes(n-m)}\right] = \begin{bmatrix} 0 & -\phi_r^0 \\ \psi_r^0 & 0 \end{bmatrix}, \\ &\widehat{B}^\dagger|_{\mathrm{qpf}} = \begin{bmatrix} & \ddots & & & & \ddots \\ & B_1^\dagger & B_0^\dagger & B_{-1}^\dagger & & \\ & & B_1^\dagger & B_0^\dagger & B_{-1}^\dagger & \\ & & & B_1^\dagger & B_0^\dagger & B_{-1}^\dagger \\ \ddots & & & & \ddots & \end{bmatrix}_{\mathrm{qpf}}, \end{aligned}\right\} \tag{3.34}$$

and h.c. Using these, we rewrite equation (3.33) as

$$\left.\begin{aligned} &\mathrm{i}\partial_\epsilon \widehat{W} = \hat{g}^0\widehat{B}^\dagger\big|_{\mathrm{qpf}}\hat{g}^{0\dagger} = \textstyle\sum_{r\in\mathbb{Z}}(\mathrm{i}\partial_\epsilon W_r)z^r, \\ &\mathrm{i}\partial_\epsilon W_r = \textstyle\sum_{k,l\in\mathbb{Z}} g_k^0 B_{k-l-r}^\dagger\big|_{\mathrm{qpf}} g_l^{0\dagger} = \sum_{k,l\in\mathbb{Z}} g_k^0 \begin{bmatrix} 0 & -\phi_{k-l-r}^0 \\ \psi_{k-l-r}^0 & 0 \end{bmatrix} g_l^{0\dagger}, \end{aligned}\right\} \tag{3.35}$$

and h.c.

Let $a(\bar{a})$ and $\mathrm{i}(\bar{i})$ be $1,\ldots,m$ hole states and $m+1,\ldots,n$ particle states of the QPF, respectively. Substituting the second equation of (3.35) into (3.8), for $r\neq 0$ we get

$$\mathrm{i}\partial_\epsilon(\mathcal{F}_r)_{\alpha\beta}=[\alpha\beta|\gamma\delta]\sum_{k,l\in\mathbb{Z}}\left\{\left(g_k^0\right)_{\delta i}\left(g_l^{0\dagger}\right)_{a\gamma}\left(\psi_{k-l-r}^0\right)_{ia}-\left(g_k^0\right)_{\delta a}\left(g_l^{0\dagger}\right)_{i\gamma}\left(\phi_{k-l-r}^0\right)_{ai}\right\}. \tag{3.36}$$

Thus, we can reach to the desired form of the equation, part of the formal RPA equation on the infinite-dimensional Grassmannian (3.29),

$$\begin{aligned}
i(\hat{g}^{0\dagger}\cdot\partial_\epsilon\mathcal{F}\cdot\hat{g}^0)_r&=\sum_{k,l\in\mathbb{Z}}g_k^{0\dagger}\cdot\mathrm{i}\partial_\epsilon\mathcal{F}_{k-l+r}\cdot g_l^0\\
&=\textstyle\sum_{k,l\in\mathbb{Z},\bar{k},\bar{l}\in\mathbb{Z}}\\
&\quad\times\left[\begin{array}{l}
\begin{bmatrix}kl\\ab\end{bmatrix}|\boldsymbol{F}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl\\ab\end{bmatrix}|\overline{\boldsymbol{F}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}},\\
\begin{bmatrix}kl\\ia\end{bmatrix}|\boldsymbol{D}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl\\ia\end{bmatrix}|\overline{\boldsymbol{D}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}},\\
\begin{bmatrix}kl\\ai\end{bmatrix}|\boldsymbol{D}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl\\ai\end{bmatrix}|\overline{\boldsymbol{D}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}}\\
\begin{bmatrix}kl\\ij\end{bmatrix}|\boldsymbol{F}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl\\ij\end{bmatrix}|\overline{\boldsymbol{F}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}}
\end{array}\right].
\end{aligned}\tag{3.37}$$

Substituting the above result into (3.29), we can derive the formal RPA equation on F_∞. Various types of the above matrix elements are expressed in the following forms:

$$\left.\begin{array}{l}
\left\{\begin{array}{l}
\begin{bmatrix}kl\\ab\end{bmatrix}|\boldsymbol{F}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{a\alpha}\left(g_l^0\right)_{\beta b}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{i}}\left(g_{\bar{l}}^{0\dagger}\right)_{\bar{a}\gamma},\\
\begin{bmatrix}kl\\ab\end{bmatrix}|\overline{\boldsymbol{F}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{a\alpha}\left(g_l^0\right)_{\beta b}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{a}}\left(g_{\bar{l}}^{0\dagger}\right)_{\bar{i}\gamma},
\end{array}\right.\\
\left\{\begin{array}{l}
\begin{bmatrix}kl\\ij\end{bmatrix}|\boldsymbol{F}|.\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{i\alpha}\left(g_l^0\right)_{\beta j}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{i}}\left(g_{\bar{l}}^{0\dagger}\right)_{\bar{a}\gamma},\\
\begin{bmatrix}kl\\ij\end{bmatrix}|\overline{\boldsymbol{F}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{i\alpha}\left(g_l^0\right)_{\beta j}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{a}}\left(g_{\bar{l}}^{0\dagger}\right)_{\bar{i}\gamma},
\end{array}\right.\\
\left\{\begin{array}{l}
\begin{bmatrix}kl\\ia\end{bmatrix}|\boldsymbol{D}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{i\alpha}\left(g_l^0\right)_{\beta a}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{i}}\left(g_{\bar{l}}^0\right)_{\bar{a}\gamma},\\
\begin{bmatrix}kl\\ia\end{bmatrix}|\overline{\boldsymbol{D}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{i\alpha}\left(g_l^0\right)_{\beta a}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{a}}\left(g_{\bar{l}}^0\right)_{\bar{i}\gamma},
\end{array}\right.\\
\left\{\begin{array}{l}
\begin{bmatrix}kl\\ai\end{bmatrix}|\boldsymbol{D}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{i}\bar{a}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{a\alpha}\left(g_l^0\right)_{\beta i}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{i}}\left(g_{\bar{l}}^0\right)_{\bar{a}\gamma},\\
\begin{bmatrix}kl\\ai\end{bmatrix}|\overline{\boldsymbol{D}}|\begin{bmatrix}\bar{k}\bar{l}\\\bar{a}\bar{i}\end{bmatrix}\stackrel{d}{=}\left(g_k^{0\dagger}\right)_{a\alpha}\left(g_l^0\right)_{\beta i}[\alpha\beta|\gamma\delta]\left(g_{\bar{k}}^0\right)_{\delta\bar{a}}\left(g_{\bar{l}}^0\right)_{\bar{i}\gamma}.
\end{array}\right.
\end{array}\right\}\tag{3.38}$$

Finally, we summarize equations to determine the collective submanifold and motion in the following forms:

The canonicity condition (3.15):

$$\langle\hat{g}^0|\partial_{\left(\substack{\epsilon\\\epsilon^*}\right)}|\hat{g}^0\rangle=\sum_{\alpha=1}^{m}\sum_{s\in\mathbb{Z}}\left(g_s^{0\dagger}\partial_{\left(\substack{\epsilon\\\epsilon^*}\right)}g_s^0\right)_{\alpha\alpha}=\frac{1}{2}\begin{pmatrix}\epsilon^*\\-\epsilon\end{pmatrix}. \tag{3.39}$$

The formal RPA equation (3.29):

$$\left.\begin{aligned}&\omega_c\Gamma\{\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\} + \mathrm{i}\partial_\epsilon\hat{\epsilon} - [\hat{\epsilon}, \theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}] - i\hat{g}^{0\dagger}\partial_\epsilon\mathcal{F}(\hat{g}^0)\hat{g}^0 = 0,\\ &\hat{g} = \hat{g}^0(\epsilon,\epsilon^*)\,\mathrm{e}^{-\mathrm{i}\hat{\epsilon}(\epsilon,\epsilon^*)t}.\end{aligned}\right\} \qquad (3.40)$$

Through constructions of the TDHF theory and the formal RPA equation on F_∞, the following becomes apparent: the ordinary perturbative method for collective variables η and η^* [10] is involved in the method of construction of the TDHF theory on the affine Kac–Moody algebra if we restrict ourselves to $\widehat{su_n}$. When η and η^* are represented as $\eta = \sqrt{\Omega}\,\mathrm{e}^{\mathrm{i}\varphi}$ and $\eta^* = \sqrt{\Omega}\,\mathrm{e}^{-\mathrm{i}\varphi}$, we can always express $\gamma(\eta,\eta^*) = \sum_{r,s\in\mathbb{Z}}\bar{\gamma}_{r,s}\eta^{*r}\eta^s = \sum_r \gamma_r z^r$ on the Lie algebra if we put $z = \mathrm{e}^{\mathrm{i}\varphi}$. This means that the infinite-dimensional Lie algebra in the SCF theory is introduced in a natural way and is useful to study various motions of fermion many-body systems.

4. Summary and concluding remarks

The formal RPA equation has been provided as a tool for truncating a collective submanifold with only one normal mode out of an infinite-dimensional Grassmannian. We have given a simple geometrical interpretation for the formal RPA equation. The collective submanifold is interpreted as a rotator on a curved surface in the infinite-dimensional Grassmannian. In F_∞, to study motions of finite fermion systems, it is manifestly natural and useful to introduce the infinite-dimensional Lie algebra arising from the anti-commutation relation between fermions. In order to discuss the relation between TDHF theory and soliton theory, we have given expressions for TDHF theory on the τ-functional space along soliton theory. From the *loop* group viewpoint and with a clearer physical picture, we have proposed a method of description of particle and collective motions in SCF theory on F_∞ in relation to an iso-spectral equation in soliton theory. Then, SCF theory on F_∞ may be regarded as soliton theory in the sense that it is based on the infinite-dimensional Grassmannian and may describe dynamics on an infinite set of *real fermion-harmonic oscillators*. On the other hand, soliton theory describes dynamics on the *complex fermion-harmonic oscillators*. It is one of the most challenging problems to extend the real space $\widehat{su_n}$ to the complex space $\widehat{sl_n}$ in TDHF theory on F_∞ together with removal of the restriction $|z| = 1$. Concerning the construction of soliton theory on multi-dimensional space [13, 14], we have an interesting future problem: to extend the Plücker relation (Hirota's form) with only one circle to the case of multi-circles such that SCF method on F_∞ can describe the dynamics of fermion systems in terms of multi-RPA bosons.

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Appendix A. Reconstruction of the p–h subgroup on F_∞

According to the particle–hole (p–h) formalism, we consider reconstruction of the TDHF equation with an n almost periodic sequence on an infinite-dimensional Fock space. We call unoccupied and occupied states of the infinite-dimensional fermions for the highest weight vector $|m\rangle$, particles and holes, respectively,

$$\left.\begin{array}{lll} \text{particle state:} & \psi_i, \psi_j, \ldots; \psi_i^*, \psi_j^*, \ldots \; i = nr + \alpha, & \begin{pmatrix} r = 0, \; \alpha = m+1, \ldots, n \\ r > 0, \; \alpha = 1, \ldots, n \end{pmatrix} \\ \text{hole state:} & \psi_a, \psi_b, \ldots; \psi_a^*, \psi_b^*, \ldots \; a = nr + \alpha. & \begin{pmatrix} r = 0, \; \alpha = 1, \ldots, m \\ r < 0, \; \alpha = 1, \ldots, n \end{pmatrix} \end{array}\right\} \tag{A.1}$$

The particle pair: $\psi_i \psi_j^*$:and the hole one: $\psi_a \psi_b^*$:are closed under the Lie multiplication as

$$\left.\begin{array}{l} [: \psi_i \psi_j^* :, : \psi_k \psi_l^* :] = \delta_{jk} : \psi_i \psi_l^* : - \delta_{il} : \psi_k \psi_j^* :, \\ [: \psi_a \psi_b^* :, : \psi_c \psi_d^* :] = \delta_{bc} : \psi_a \psi_d^* : - \delta_{ab} : \psi_c \psi_b^* : - \delta_{ad} \delta_{cb} (b \leqslant 0). \end{array}\right\} \tag{A.2}$$

We decompose the generator of $\widehat{su_n}$ (3.1) into two components each of which unchanges and changes the number of particles and holes, respectively, in the following forms:

$$X_{\gamma'} = \zeta_{ia} : \psi_i \psi_a^* : - \zeta_{ia}^* : \psi_a \psi_i^* : + \mathbb{C}', \qquad X_{\gamma''} = \bar{\eta}_{ij} : \psi_i \psi_j^* : + \eta_{ab} : \psi_a \psi_b^* : + \mathbb{C}'', \tag{A.3}$$

$$\zeta = \begin{bmatrix} \zeta_{-N}^{J,n} & \cdots & \zeta_{-2}^{J,n} & \zeta_{-1}^{J,n} & \zeta_0^{J,I} \\ & \zeta_{-N}^{n,n} & & \zeta_{-2}^{n,n} & \zeta_{-1}^{n,I} \\ & & \ddots & \vdots & \zeta_{-2}^{n,I} \\ & & & \zeta_{-N}^{n,n} & \vdots \\ & & & & \zeta_{-N}^{n,I} \end{bmatrix}, \qquad \mathrm{Tr}(\zeta_r) = 0, \tag{A.4}$$

$$\eta = \begin{bmatrix} \ddots & & & & \ddots & & \\ & \eta_0^{n,n} & \cdots & \cdots & \cdots & \eta_N^{n,n} & \\ & \vdots & \ddots & & & \vdots & \eta_N^{n,I} \\ & \vdots & & \ddots & & \vdots & \vdots \\ \ddots & \vdots & & & \eta_0^{n,n} & \eta_1^{n,n} & \vdots \\ & \eta_{-N}^{n,n} & \cdots & \cdots & \eta_{-1}^{n,n} & \eta_0^{n,n} & \eta_1^{n,I} \\ & & \eta_{-N}^{I,n} & \cdots & \cdots & \eta_{-1}^{I,n} & \eta_0^{I,I} \end{bmatrix}, \qquad \bar{\eta} = \begin{bmatrix} \bar{\eta}_0^{J,J} & \bar{\eta}_1^{J,n} & \cdots & \cdots & \bar{\eta}_N^{J,n} & & \\ \bar{\eta}_{-1}^{n,J} & \bar{\eta}_0^{n,n} & \cdots & \cdots & \cdots & \bar{\eta}_N^{n,n} & \\ \vdots & \vdots & \ddots & & & \vdots & \ddots \\ \vdots & \vdots & & \ddots & & \vdots & \\ \bar{\eta}_{-N}^{n,J} & \vdots & & & \ddots & \vdots & \\ & \bar{\eta}_{-N}^{n,n} & \cdots & \cdots & \cdots & \bar{\eta}_0^{n,n} & \\ & & \ddots & & & & \ddots \end{bmatrix}, \tag{A.5}$$

where both $\mathbb{C}'$ and $\mathbb{C}''$ are pure imaginary and $\mathrm{Tr}(\eta_r) = \mathrm{Tr}(\bar{\eta}_r) = 0$. Indices I and J run $1, \ldots, m$ and $m+1, \ldots, n$. Any matrices $M_r^{I,J}$ are $(I \times J)$-dimensional entries in any M_r.

By the same way as the derivation of equation (3.4), we obtain

$$\mathrm{e}^{X_{\gamma''}} \psi_a \, \mathrm{e}^{-X_{\gamma''}} = \psi_b \hat{w}_{ba}, \qquad \mathrm{e}^{X_{\gamma''}} \psi_i \, \mathrm{e}^{-X_{\gamma''}} = \psi_j \hat{\bar{w}}_{ij}, \tag{A.6}$$

where $\hat{w} = (\hat{w}_{ab})$ and $\hat{\bar{w}} = (\hat{\bar{w}}_{ij})$ are infinite-dimensional matrices on the hole and particle space but have not a periodic sequence with period n as discussed in section 3. The action of $\mathrm{e}^{X_{\gamma''}}$ for $|m\rangle$ leaves $|m\rangle$ invariant under the phase equivalence relation

$$\mathrm{e}^{X_{\gamma''}} |m\rangle = \mathrm{e}^{\mathrm{i}\mathbb{C}''} \det \hat{w} |m\rangle = \mathrm{e}^{\mathrm{i}\delta} |m\rangle, \qquad \delta \overset{d}{=} \sum_{a=1}^{m} (\eta_0)_{aa} + \mathbb{C}''. \tag{A.7}$$

Consider the ph-type generator $X_{\gamma'}$. Following the appendix A of [5], regarding ζ (A.4) as an embedded form into a $(nN + J) \times (nN + I)$-dimensional matrix, we obtain

$$\mathrm{e}^{X_{\gamma'}} \psi_a \, \mathrm{e}^{-X_{\gamma'}} = \psi_b \hat{C}(\zeta)_{ba} + \psi_j \hat{S}(\zeta)_{ja}, \qquad \mathrm{e}^{X_{\gamma'}} \psi_i \, \mathrm{e}^{-X_{\gamma'}} = \psi_j \hat{\bar{C}}(\zeta)_{ji} - \psi_b \hat{S}^{\dagger}(\zeta)_{bi}. \tag{A.8}$$

The $\hat{S}(\zeta)$, $\hat{C}(\zeta)$ and $\hat{\hat{C}}(\zeta)$ are $(nN+J)\times(nN+I)$-, $(nN+I)\times(nN+I)$- and $(nN+J)\times(nN+J)$-dimensional triangular matrix functions defined as

$$\left.\begin{aligned}
\hat{S}(\zeta) &= \textstyle\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}\zeta(\zeta^\dagger\zeta)^k, \\
\hat{C}(\zeta) &= I_{nN+I} + \textstyle\sum_{k=1}^{\infty}(-1)^k \frac{1}{(2k)!}(\zeta^\dagger\zeta)^k = \hat{C}^\dagger(\zeta), \\
\hat{\hat{C}}(\zeta) &= I_{nN+J} + \textstyle\sum_{k=1}^{\infty}(-1)^k \frac{1}{(2k)!}(\zeta\zeta^\dagger)^k = \hat{\hat{C}}^\dagger(\zeta),
\end{aligned}\right\} \tag{A.9}$$

where I_{Nn+I} is the unit matrix of $(nN+I)$-dimension. Since $\zeta^\dagger\zeta$ and $\zeta\zeta^\dagger$ are the positive Hermitian matrices, their matrices have properties analogous to the usual triangular ones

$$\hat{C}(\zeta)^2 + \hat{S}^\dagger(\zeta)\hat{S}(\zeta) = I_{nN+I}, \qquad \hat{\hat{C}}(\zeta)^2 + \hat{S}(\zeta)\hat{S}^\dagger(\zeta) = I_{nN+J}, \qquad \hat{S}(\zeta)\hat{C}(\zeta) = \hat{\hat{C}}(\zeta)\hat{S}(\zeta). \tag{A.10}$$

Using a $\{(2nN+I+J)\times(2nN+I+J)\}$-dimensional matrix $\hat{g}^{\rm sub}(\zeta)$, we define an $\widehat{su_n}\hat{g}(\zeta)$

$$\hat{g}(\zeta) = \begin{bmatrix} \ddots & & & & \\ & I_n & & & \\ & & \hat{g}^{\rm sub}(\zeta) & & \\ & & & I_n & \\ & & & & \ddots \end{bmatrix}, \qquad \hat{g}^{\rm sub}(\zeta) = \begin{bmatrix} \hat{C}(\zeta) & -\hat{S}^\dagger(\zeta) \\ \hat{S}(\zeta) & \hat{\hat{C}}(\zeta) \end{bmatrix}. \tag{A.11}$$

Here, for the $\hat{g}(\zeta)$ we use notation below

$$\hat{g}_{nr+\alpha,ns+\beta}(\zeta) = \begin{cases} \hat{g}^{\rm sub}_{nr+\alpha,ns+\beta}(\zeta), & |r| \text{ and } |s| \leqslant N \\ \delta_{rs}\delta_{\alpha\beta}, & \text{otherwise} \end{cases} \tag{A.12}$$

together with its Hermitian conjugate where $\hat{g}(\zeta)\hat{g}^\dagger(\zeta) = \hat{g}^\dagger(\zeta)\hat{g}(\zeta) = \hat{I}$. Thus, the $\hat{g}(\zeta)$ is an exact $\widehat{su_n}$ matrix and

$$\hat{g}^{\rm sub}_{nr+\alpha,ns+\beta}(\zeta) = \{g_{s-r}(r,s;\zeta)\}_{\alpha\beta}, \tag{A.13}$$

together with the same relation for $\hat{g}^{sub\dagger}_{nr+\alpha,ns+\beta}(\zeta)$ and $\hat{g}^{\rm sub}(\zeta)\hat{g}^{\rm sub\dagger}(\zeta) = \hat{g}^{\rm sub\dagger}(\zeta)\hat{g}^{\rm sub}(\zeta) = I_{(2nN+I+J)}$. Then, the canonical transformation (A.8) and its Hermitian conjugation are recast, respectively, as

$$\begin{aligned}
\psi_{nr+\alpha}\{\hat{g}(\zeta)\} &= U\{\hat{g}(\zeta)\}\psi_{nr+\alpha}U^{-1}\{\hat{g}(\zeta)\} = \sum_{s\in\mathbb{Z}}\psi_{ns+\beta}\hat{g}_{ns+\beta,nr+\alpha}(\zeta) \\
&= \begin{cases} \sum_{s=-N}^{N}\psi_{ns+\beta}\hat{g}^{\rm sub}_{ns+\beta,nr+\alpha}(\zeta), & |r| \leqslant N \\ \psi_{nr+\alpha}, & \text{otherwise} \end{cases}
\end{aligned} \tag{A.14}$$

together with its Hermitian conjugate. If a constraint on $\hat{g}^{\rm sub}(\zeta)$ via the Lie elements ζ is a form of (A.4), its structure can be determined up to the phase of subgroup orbit, $U\{\hat{g}(\zeta)\hat{g}(w)\}|m\rangle[=U\{\hat{g}(\zeta)\}\,{\rm e}^{{\rm i}\delta}|m\rangle]$.

Appendix B. Embedding of SCF Hamiltonian into F_∞

We will embed canonical transformation into F_∞. In the case of $\widetilde{gl_n}$, using $\gamma_r^\dagger = -\gamma_{-r}$ we have a representation for algebra $\tau(\gamma(z))$ and group $\tilde{g} \stackrel{d}{=} \exp[\tau(\gamma(z))]$ with a unitary condition $\tilde{g}\tilde{g}^\dagger = \tilde{g}^\dagger\tilde{g} = I$. In the conventional picture of SCF theory, the corresponding matrix $g(z) = \mathrm{e}^{\gamma(z)}$ satisfies

$$g(z)g^\dagger(z) = \sum_{r,s\in\mathbb{Z}} g_r z^r g_s^\dagger z^{-s} = \sum_{k,s\in\mathbb{Z}} g_{s+k} g_s^\dagger z^k = \delta_{k,0}\cdot I_n. \tag{B.1}$$

This means $\sum_{s\in\mathbb{Z}} g_s g_s^\dagger = I_n$ and $\sum_{s\in\mathbb{Z}} g_{s+k} g_s^\dagger = 0 (k \neq 0)$. The corresponding representation for $\gamma(z)(= -\gamma^\dagger(z))$ is given as

$$\tau\{\gamma(z)\} = \sum_{r,s\in\mathbb{Z}\alpha,\beta=1,\ldots,n} (\gamma_r)_{\alpha\beta}\psi_{n(s-r)+\alpha}\psi^*_{ns+\beta}. \tag{B.2}$$

Defining $\Psi_r \equiv (\psi_{nr+1}, \psi_{nr+2}, \ldots, \psi_{nr+n})$, canonical transformations on F_∞ are given by

$$\{\ldots, \Psi_{-1}(\tilde{g}), \Psi_0(\tilde{g}), \Psi_1(\tilde{g}), \ldots\} = \{\ldots, \Psi_{-1}, \Psi_0, \Psi_1, \ldots\}\,\tilde{g}, \tag{B.3}$$

and

$$\psi_{nr+\alpha}(\tilde{g}) = U(\tilde{g})\psi_{nr+\alpha}U(\tilde{g})^{-1} = \sum_{s\in\mathbb{Z}}\sum_{\beta=1}^{n} \psi_{n(r-s)+\beta}(g_s)_{\beta\alpha}, \tag{B.4}$$

together with its Hermitian conjugate. On the other hand, in the conventional picture of SCF, the corresponding creation operator is expressed as

$$c_\alpha^\dagger\{g(z)\} = U\{g(z)\}c_\alpha^\dagger U^{-1}\{g(z)\} = \sum_{\beta=1}^{n} c_\beta^\dagger g_{\beta\alpha}(z) = \sum_{s\in\mathbb{Z}}\sum_{\beta=1}^{n} c_\beta^\dagger z^s (g_s)_{\beta\alpha}, \tag{B.5}$$

together with its Hermitian conjugate. Then, multiplying z^{-r} and z^r to the above equations, we get

$$\left.\begin{array}{l} z^{-r}\cdot c_\alpha^\dagger\{g(z)\} = \sum_{s\in\mathbb{Z}} z^{-(r-s)} c_\beta^\dagger\cdot(g_s)_{\beta\alpha}, \\ c_\alpha\{g(z)\}\cdot z^r = \sum_{s\in\mathbb{Z}} c_\beta z^{(r-s)}\cdot(g_s)^*_{\beta\alpha}. \end{array}\right\} \tag{B.6}$$

Putting $z^{-r}c_\alpha^\dagger\{g(z)\} = \psi_{nr+\alpha}(\tilde{g})$ and $z^{-r}c_\alpha^\dagger = \psi_{nr+\alpha}$, we get the relation (B.4).

We embed the density matrix $W_{\alpha\beta}(g) = \sum_{\gamma=1}^m g_{\alpha\gamma}(g^\dagger)_{\gamma\beta}$ into F_∞ and realize it as follows:

$$\left.\begin{array}{l} W_{\alpha\beta}\{g(z)\} = \sum_{\gamma=1}^m \sum_{r,s\in\mathbb{Z}} (g_r)_{\alpha\gamma}\left(g_s^\dagger\right)_{\gamma\beta} z^{r-s} = \sum_{k\in\mathbb{Z}} (W_k)_{\alpha\beta} z^k (r-s=k), \\ (W_k)_{\alpha\beta} = \sum_{\gamma=1}^m \sum_{s\in\mathbb{Z}} (g_{s+k})_{\alpha\gamma}\left(g_s^\dagger\right)_{\gamma\beta} = \sum_{\gamma=1}^m \sum_{r\in\mathbb{Z}} (g_r)_{\alpha\gamma}\left(g_{r-k}^\dagger\right)_{\gamma\beta}. \end{array}\right\} \tag{B.7}$$

In F_∞, we define the density matrix as $W^f_{nr+\alpha,ns+\beta} \stackrel{d}{=} \langle m|U^\dagger(\tilde{g})\psi_{ns+\beta}\psi^*_{nr+\alpha}U(\tilde{g})|m\rangle$ where the reference and perfect vacuums are defined in section 3. Using the canonical transformation (B.4), the density matrix transforms to

$$W^f_{nr+\alpha,ns+\beta} = \sum_{\alpha',\beta'}\sum_{r',s'} (g_{r'})_{\alpha\alpha'}\left(g_{s'}^\dagger\right)_{\beta'\beta}\langle m|\psi_{n(s-s')+\beta'}\psi^*_{n(r-r')+\alpha'}|m\rangle. \tag{B.8}$$

Finally, we embed the usual SCF Hamiltonian $H_{\mathrm{HF}}[W]$ into F_∞ as

$$H_{\mathrm{HF}}[W] = \sum_{k\in\mathbb{Z}} \{h_{\alpha\beta}\delta_{k,0} + [\alpha\beta|\gamma\delta](W_k)_{\delta\gamma}\} z^k c_\beta^\dagger c_\alpha, \tag{B.9}$$

and

$$\tau(H_{\mathrm{HF}}[W]) = \sum_{k\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} \{h_{\alpha\beta}\delta_{k,0} + [\alpha\beta|\delta\gamma](W_k)_{\delta\gamma}\}\psi_{n(s-k)+\beta}\psi^*_{ns+\alpha}. \tag{B.10}$$

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